Massive higher spins in $d = 3$ unfolded

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Abstract

In this paper we construct an unfolded formulation for the massive bosonic higher spins in three dimensions as well as for their partially massless limit of the maximal depth. We begin with the equations for the one-forms coming from the frame-like gauge invariant Lagrangians for such fields and then we supplement them with the infinite number of equations for the appropriately chosen set of zero-forms.

Introduction

In the recent paper [1] new linear unfolded systems of equations [2, 3] (see also [4] and references therein) for the set of bosonic zero-forms in the three dimensional anti-de Sitter space were investigated. They appear as one parameter deformation for the linearized critical Prokushkin-Vasiliev theory [5] and were shown to be reach enough to describe topologically massive higher spins, fractional spins and so on. In this paper we are interested in the parity even massive bosonic higher spins as well as the so called partially massless limits of the maximal depth [6, 7, 8, 9]. Recall that in three dimensions massless higher spins $s \geq 2$ as well as all partially massless ones except the one with the maximal depth do not have any physical degrees of freedom. At the same time, massive higher spins have two physical degrees of freedom (exactly as massless one in $d = 4$), while the partially massless fields with the maximal depth have just one. That is why we restrict ourselves with these two cases here.

In [1] the authors begin directly with the new set of equations for the zero-forms, then investigate their relation with the representations of anti-de Sitter group $SO(2,2)$ and look for the possibilities to supplement these zero-forms with the appropriate set of gauge potentials. In the current paper we will go just the other way round. Namely, we begin with the frame-like gauge invariant description for the massive higher spins in $d = 3$ [10, 11] (for dimensions greater than three see [9, 12]) where from the very beginning we perfectly know what these systems describe and then we supplement the equations for the gauge one-forms with the appropriate set of zero-forms and their equations. Note that exactly as the gauge invariant description of massive higher spins itself our construction works not only in anti-de Sitter space but in Minkowski and de Sitter spaces as well provided $m^2 \geq (s - 1)^2 \Lambda$.

The crucial questions for us was what is the correct set of zero-forms one has to introduce to have unfolded formulation for massive arbitrary spins. In $d \geq 4$ frame-like description for the massless spin-s field requires introduction of the physical, auxiliary and extra one-forms:

$$\Phi_\mu{}^{a(s-1)}, \quad \Omega_\mu{}^{a(s-1),b}, \quad \Sigma_\mu{}^{a(s-1),b(k)}, \quad 2 \leq k \leq s - 1$$

while for the unfolded formulation one has to introduce a whole set of zero-forms:

$$W^{a(k),b(s)}, \quad s \leq k$$

where tensor $T^{a(k),b(l)}$ corresponds to the two-raw Young tableau $Y(k,l)$. At the same time, unfolded descriptions for the spin-1 and spin-0 fields requires, correspondingly:

$$A_\mu, \quad F^{a(k),b}, \quad 1 \leq k$$

$$\varphi, \quad \pi^{a(k)}, \quad 1 \leq k$$

The main idea of the gauge invariant description for massive higher spin s is that such description can be constructed out of the massless fields with spins $s, s-1, \ldots, 0$. Similarly, in [12] the unfolded formulation for the massive higher spins in $d \geq 4$ was constructed combining all the one-forms and zero-forms necessary for the unfolding of all these massless fields.

Now let us turn to the $d = 3$. A lot of important consequences follow from the simple fact that in $d = 3$ antisymmetric second rank tensor is equivalent to the vector:

$$A^a = \varepsilon^{abc} B_{bc}$$

In particular, one can show that any mixed tensor corresponding to the Young tableau $Y(k, l)$ with $l \geq 2$ is identically zero. As a result, frame-like formalism for the massless spin-s fields with $s \geq 2$ requires physical and auxiliary fields only, while all zero forms are identically zero in agreement with the known fact that such fields do not have any physical degrees of freedom. Thus only spin-1 and spin-0 fields require zero-forms for the unfolded description. Moreover for the spin-1 case all zero-forms can be dualized into the completely symmetric traceless tensors. This restricts us with the just two sets of zero-forms, originating from the spin-1 and spin-0. It was not at all evident that the very same zero-forms may describe massive arbitrary spins but the results of [1] show that it is indeed possible.

Note that, in-principle, starting with the results of [12], setting most of the one-forms and zero-forms to zero, choosing appropriate solution and scaling for the coefficients and dualizing all the objects that are not completely symmetric in local indices, one can obtain the desired unfolded description for $d = 3$ case. But as it often happens, especially when one deals with three-dimensional theories, it is easier to straightforwardly derive such description directly in $d = 3$. Moreover, this allows us to take into account peculiarities of three-dimensional theories that have no analogues in $d \geq 4$.

The paper is organized as follows. In section 1 we begin with the rather simple but instructive example of massive spin-2 and its partially massless limit illustrating most of the features of our construction. In section 2 we consider spin-3 (also both massive as well as partially massless case) as one more concrete example illustrating how the same set of zero-forms begins to describe higher spin. At last in section 3 we gives the description for the massive field with arbitrary spin. To make our paper to be self-contained we supply three appendices which contains all necessary information on the frame-like gauge invariant description of massive bosonic higher spins (appendix A), its partially massless limits in-particular the one with the maximal depth (appendix B) and also on the partial gauge fixing (appendix C) that greatly simplifies calculations with massive arbitrary spins.

Notations and conventions. We will work in the three dimensional frame-like formalism where massless spin s is described by the physical $\Phi_\mu{}^{a_1 a_1 \dots a_{s-1}}$ and auxiliary $\Omega_\mu{}^{a_1 a_2 \dots a_{s-1}}$ one-forms. Here Greek letters denote world indices, while Latin letters denote local ones. All objects will be assumed to be completely symmetric and traceless in their local indices. To simplify formulas we will use the shorthand notation such as

$$\Phi_\mu{}^{a(k)} = \Phi_\mu{}^{a_1 a_2 \dots a_k}$$

Besides all the indices denoted by the same letter and placed on the same level will be assumed symmetrized, e.g.

$$e_\mu{}^a \eta^{a(k)} = e_\mu{}^{(a_1} \eta^{a_2 \dots a_{k+1})}$$

where we use symmetrization without any normalization factor. We will work in the (anti)-de Sitter space with arbitrary cosmological constant Λ, the frame field $e_\mu{}^a$ and the covariant derivative D_μ normalized so that

$$D_{[\mu} D_{\nu]} \eta^a = -\Lambda e_{[\mu}{}^a \eta_{\nu]}$$

2

1 Spin 2

In this section we consider a rather simple but instructive example of spin 2, illustrating almost all the general features of our construction. We begin with the partially massless case and then we will turn to the general massive one.

1.1 Partially massless case

For the frame-like gauge invariant description of the partially massless spin-2 we need two pairs of physical and auxiliary fields $(\Omega_\mu{}^a, \Phi_\mu{}^a)$ and (B^a, A_μ).[1] The free Lagrangian describing partially massless spin-2 in the three dimensional de Sitter space with positive cosmological term Λ has the form:

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\{{}^{\mu\nu}_{ab}\}\Omega_\mu{}^a\Omega_\nu{}^b - \varepsilon^{\mu\nu\alpha}\Omega_\mu{}^a D_\nu \Phi_{\alpha,a} + B^a B_a - \varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha \\
&\quad - m\varepsilon^{\mu\nu\alpha}[\Omega_{\mu,\nu}A_\alpha - B_\mu \Phi_{\nu,\alpha}]
\end{aligned} \tag{1}
$$

where $\{{}^{\mu\nu}_{ab}\} = e^\mu{}_a e^\nu{}_b - e^\nu{}_a e^\mu{}_b$ and $m^2 = \Lambda$. This Lagrangian is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta\Omega_\mu{}^a &= D_\mu\eta^a, & \delta\Phi_\mu{}^a &= D_\mu\xi^a + \varepsilon_\mu{}^{ab}\eta_b + me_\mu{}^a\xi \\
\delta B^a &= -m\eta^a, & \delta A_\mu &= D_\mu\xi + m\xi_\mu
\end{aligned} \tag{2}
$$

There exist four gauge invariant objects:

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^a &= D_{[\mu}\Omega_{\nu]}{}^a - me_{[\mu}{}^a B_{\nu]} \\
\mathcal{T}_{\mu\nu}{}^a &= D_{[\mu}\Phi_{\nu]}{}^a + \varepsilon_{[\mu}{}^{ab}\Omega_{\nu]b} + me_{[\mu}{}^a A_{\nu]} \\
\mathcal{B}_\mu{}^a &= D_\mu B^a + m\Omega_\mu{}^a \\
\mathcal{A}_{\mu\nu} &= D_{[\mu}A_{\nu]} - 2\varepsilon_{\mu\nu a}B^a - m\Phi_{[\mu,\nu]}
\end{aligned} \tag{3}
$$

Let us take the first four unfolded equations in the form:

$$
\begin{aligned}
0 &= D_{[\mu}\Omega_{\nu]}{}^a - me_{[\mu}{}^a B_{\nu]} \\
0 &= D_{[\mu}\Phi_{\nu]}{}^a + \varepsilon_{[\mu}{}^{ab}\Omega_{\nu]b} + me_{[\mu}{}^a A_{\nu]} \\
0 &= D_\mu B^a + m\Omega_\mu{}^a - B_\mu{}^a \\
0 &= D_{[\mu}A_{\nu]} - 2\varepsilon_{\mu\nu a}B^a - m\Phi_{[\mu,\nu]}
\end{aligned} \tag{4}
$$

where the zero-form B^{ab} is symmetric and traceless.[2] All these equations except the third one are already consistent while the consistency for the third one gives:

$$
\begin{aligned}
0 &= D_{[\mu}D_{\nu]}B^a + mD_{[\mu}\Omega_{\nu]}{}^a - D_{[\mu}B_{\nu]}{}^a \\
&= -\Lambda e_{[\mu}{}^a B_{\nu]} + m^2 e_{[\mu}{}^a B_{\nu]} - D_{[\mu}B_{\nu]}{}^a \\
&= D_{[\mu}B_{\nu]}{}^a
\end{aligned} \tag{5}
$$

---

[1] Recall that in $d = 4$ such field has four physical degrees of freedom, namely helicities $\pm2, \pm1$ (and this explains the set of fields introduced), while in $d = 3$ it has only one.

[2] Note once again that in $d = 3$ massless fields with spins $s \geq 2$ do not have any physical degrees of freedom, thus there is no need to introduce zero forms for them.

Taking into account that we deal with the parity even theory, we choose the following ansatz for the remaining equations ($k \geq 2$):

$$0 = D_\mu B^{a(k)} - B_\mu{}^{a(k)} + E_k[e_\mu{}^a B^{a(k-1)} - \frac{2}{(2k-1)} g^{a(2)} B_\mu{}^{a(k-2)}], \quad B_2 = 0 \qquad (6)$$

where all $B^{a(k)}$ are symmetric, traceless and gauge invariant. Consistency for with ansatz requires:

$$\frac{(2k+3)}{(2k+1)} E_{k+1} - E_k - \Lambda = 0 \qquad (7)$$

and we obtain the solution:

$$E_k = \frac{(k^2 - 4)}{(2k+1)} \Lambda \qquad (8)$$

1.2 General massive case

The frame-like gauge invariant description for the general massive spin-2 requires also a pair (π^a, φ). The free Lagrangian looks like:

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2} \{ {}^{\mu\nu}_{ab} \} \Omega_\mu{}^a \Omega_\nu{}^b - \varepsilon^{\mu\nu\alpha} \Omega_\mu{}^a D_\nu \Phi_{\alpha,a} + B_a{}^2 - \varepsilon^{\mu\nu\alpha} B_\mu D_\nu A_\alpha - \pi_a{}^2 + \pi^\mu D_\mu \varphi \\
&\quad - m \varepsilon^{\mu\nu\alpha} [\Omega_{\mu\nu} A_\alpha - B_\mu \Phi_{\nu\alpha}] + 2M\pi^\mu A_\mu \\
&\quad + \frac{M^2}{2} \{ {}^{\mu\nu}_{ab} \} \Phi_\mu{}^a \Phi_\nu{}^b + mM\Phi\varphi + \frac{3m^2}{4} \varphi^2
\end{aligned}
\qquad (9)
$$

where $M^2 = m^2 - \Lambda$. Note that such description works in anti-De Sitter, Minkowski and De Sitter space provided $m^2 \geq \Lambda$. This Lagrangian is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta\Omega_\mu{}^a &= D_\mu \eta^a + M^2 \varepsilon_\mu{}^{ab} \xi_b \\
\delta\Phi_\mu{}^a &= D_\mu \xi^a + \varepsilon_\mu{}^{ab} \eta_b + m e_\mu{}^a \xi \\
\delta A_\mu &= D_\mu \xi + m\xi_\mu, \qquad \delta B^a = -m\eta^a \\
\delta\varphi &= -2M\xi, \qquad \delta\pi^a = mM\xi^a
\end{aligned}
\qquad (10)
$$

Correspondingly, in this case one can construct six gauge invariant objects:

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^a &= D_{[\mu}\Omega_{\nu]}{}^a - m e_{[\mu}{}^a B_{\nu]} + M^2 \varepsilon_{[\mu}{}^{ab} \Phi_{\nu],b} - Mm\varepsilon_{\mu\nu}{}^a \varphi \\
\mathcal{T}_{\mu\nu}{}^a &= D_{[\mu}\Phi_{\nu]}{}^a + \varepsilon_{[\mu}{}^{ab} \Omega_{\nu],b} + m e_{[\mu}{}^a A_{\nu]} \\
\mathcal{B}_\mu{}^a &= D_\mu B^a + m\Omega_\mu{}^a - M\varepsilon_\mu{}^{ab} \pi_b \\
\mathcal{A}_{\mu\nu} &= D_{[\mu}A_{\nu]} - 2\varepsilon_{\mu\nu a} B^a - m\Phi_{[\mu,\nu]} \\
\Pi_\mu{}^a &= D_\mu \pi^a - M\varepsilon_\mu{}^{ab} B_b - Mm\Phi_\mu{}^a - \frac{m^2}{2} e_\mu{}^a \varphi \\
\Phi_\mu &= D_\mu \varphi - 2\pi_\mu + 2MA_\mu
\end{aligned}
\qquad (11)
$$

Thus for the first six unfolded equations we take

$$\mathcal{F}_{\mu\nu}{}^a = 0, \qquad \mathcal{T}_{\mu\nu}{}^a = 0, \qquad \mathcal{A}_{\mu\nu} = 0, \qquad \Phi_\mu = 0 \qquad (12)$$

which are already consistent as well as:

$$0 = D_\mu B^a + m\Omega_\mu{}^a - M\varepsilon_\mu{}^{ab}\pi_b - B_\mu{}^a$$

$$0 = D_\mu\pi^a - M\varepsilon_\mu{}^{ab}B_b - Mm\Phi_\mu{}^a - \frac{m^2}{2}e_\mu{}^a\varphi - \pi_\mu{}^a \tag{13}$$

where the zero-forms B^{ab} and π^{ab} are symmetric and traceless.[3] Their consistency requires:

$$D_{[\mu}B_{\nu]}{}^a = M\varepsilon_{[\mu}{}^{ab}\pi_{\nu],b}$$

$$D_{[\mu}\pi_{\nu]}{}^a = M\varepsilon_{[\mu}{}^{ab}B_{\nu],b} \tag{14}$$

So we introduce the following general ansatz for the remaining equations $(k \geq 2)$:[4]

$$\begin{aligned}
0 = & \ D_\mu B^{a(k)} - B_\mu{}^{a(k)} + A_k\varepsilon_{\mu b}{}^a\pi^{a(k-1)b} \\
& + E_k[e_\mu{}^a B^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}B_\mu{}^{a(k-2)}] \\
0 = & \ D_\mu\pi^{a(k)} - \pi_\mu{}^{a(k)} + C_k\varepsilon_{\mu b}{}^a B^{a(k-1)b} \\
& + D_k[e_\mu{}^a\pi^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\pi_\mu{}^{a(k-2)}]
\end{aligned} \tag{15}$$

where

$$A_2 = C_2 = \frac{M}{3}, \qquad E_2 = D_2 = 0$$

Consistency for these equations requires:

$$A_k = C_k, \qquad E_k = D_k, \qquad A_{k+1} = \frac{k}{(k+2)}A_k \tag{16}$$

$$B_{k+1} = \frac{(2k+1)}{(2k+3)}[B_k + A_k{}^2 + \Lambda] \tag{17}$$

These relations can be easily solved and give the solution:

$$A_k = \frac{2M}{k(k+1)}, \qquad E_k = \frac{(k^2-4)}{(2k+1)}[\frac{M^2}{k^2} + \Lambda] \tag{18}$$

1.3 Partial gauge fixing

It is rather well known [13, 14] (see also [15] and references therein) that for the massless higher spin fields in three dimensional anti-de Sitter space one can use a separation of variables that greatly simplifies all calculations. In this subsection we will show that such

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[3]Note that for the spin-2 case the frame-like formalism in $d = 3$ requires essentially the same (up to dualization) set of fields as in $d \geq 4$, the main difference being the absence of the Weyl zero-form. As a result the structure of equations (11)-(13) is the same as that of equations (3.53)-(3.58) in [12].

[4]This ansatz is essentially the same as in [1]. Note that the zero-forms B and π here have opposite parities, so this ansatz does not break parity. Recall that the whole family of the models with matrix coefficients μ given in [1] contains both parity odd and parity even ones. Note also that seemingly difference in some coefficients is related with different conventions on symmetrization.

separation is possible for the massive spin-2 as well (for an arbitrary spin case see appendix C) provided one uses partial gauge fixing removing the scalar field. Moreover, such procedure works not only in anti-de Sitter, but in Minkowski and de Sitter spaces as well, provided $m^2 > \Lambda$.

Let us partially fix the gauge setting $\varphi = 0$, solve the corresponding constraint $\Phi_\mu = 0 \Rightarrow A_\mu = \pi_\mu/M$ and change the normalization $\pi^a \Rightarrow M\pi^a$ (because now this field will play the role of physical field and not that of the auxiliary one). The resulting Lagrangian will take the form:

$$
\begin{aligned}
\mathcal{L} = {} & \frac{1}{2} \{ {}^{\mu\nu}_{ab} \} \Omega_\mu{}^a \Omega_\nu{}^b - \varepsilon^{\mu\nu\alpha} \Omega_\mu{}^a D_\nu \Phi_{\alpha,a} + B^a B_a - \varepsilon^{\mu\nu\alpha} B_\mu D_\nu \pi_\alpha \\
& - m\varepsilon^{\mu\nu\alpha} \Omega_{\mu,\nu} \pi_\alpha + m\varepsilon^{\mu\nu\alpha} B_\mu \Phi_{\nu,\alpha} + \frac{M^2}{2} \{ {}^{\mu\nu}_{ab} \} \Phi_\mu{}^a \Phi_\nu{}^b + M^2 \pi^a \pi_a
\end{aligned}
\tag{19}
$$

This Lagrangian is still invariant under the two remaining gauge transformations:

$$
\begin{aligned}
\delta\Omega_\mu{}^a &= D_\mu \eta^a + M^2 \varepsilon_\mu{}^{ab} \xi_b \\
\delta\Phi_\mu{}^a &= D_\mu \xi^a + \varepsilon_\mu{}^{ab} \eta_b \\
\delta B^a &= -m\eta^a, \qquad \delta\pi^a = m\xi^a
\end{aligned}
\tag{20}
$$

Let us introduce the new variables:

$$
\hat{\Omega}_\mu{}^a = \Omega_\mu{}^a + M\Phi_\mu{}^a, \qquad \hat{\Phi}_\mu{}^a = \Omega_\mu{}^a - M\Phi_\mu{}^a
$$

$$
\hat{B}^a = B^a - M\pi^a, \qquad \hat{\pi}^a = B^a + M\pi^a
$$

Then the Lagrangian decompose into two independent parts:

$$
\mathcal{L} = \frac{1}{4M} [\mathcal{L}(\hat{\Omega}, \hat{B}) - \mathcal{L}(\hat{\Phi}, \hat{\pi})]
$$

where, for example,

$$
\begin{aligned}
\mathcal{L}(\hat{\Omega}, \hat{B}) = {} & 2M \{ {}^{\mu\nu}_{ab} \} \hat{\Omega}_\mu{}^a \hat{\Omega}_\nu{}^b - \varepsilon^{\mu\nu\alpha} \hat{\Omega}_\mu{}^a D_\nu \hat{\Omega}_{\alpha,a} + 2M\hat{B}^a \hat{B}_a + \varepsilon^{\mu\nu\alpha} \hat{B}_\mu D_\nu \hat{B}_\alpha \\
& + 2m\varepsilon^{\mu\nu\alpha} \hat{\Omega}_{\mu,\nu} \hat{B}_\alpha
\end{aligned}
\tag{21}
$$

This Lagrangian has only one gauge symmetry, namely:

$$
\delta\hat{\Omega}_\mu{}^a = D_\mu \hat{\eta}^a + M\varepsilon_\mu{}^{ab} \hat{\eta}_b, \qquad \delta\hat{B}^a = -m\hat{\eta}^a, \qquad \hat{\eta}^a = \eta^a + M\xi^a
\tag{22}
$$

Correspondingly, there exist two gauge invariant objects:

$$
\begin{aligned}
\hat{\mathcal{F}}_{\mu\nu}{}^a &= D_{[\mu} \hat{\Omega}_{\nu]}{}^a - me_{[\mu}{}^a \hat{B}_{\nu]} + M\varepsilon_{[\mu}{}^{ab} \hat{\Omega}_{\nu],b} \\
\hat{\mathcal{B}}_\mu{}^a &= D_\mu \hat{B}^a + m\hat{\Omega}_\mu{}^a + M\varepsilon_\mu{}^{ab} \hat{B}_b
\end{aligned}
\tag{23}
$$

As the first two unfolded equations we take:

$$
\begin{aligned}
0 &= D_{[\mu} \hat{\Omega}_{\nu]} - me_{[\mu} \hat{B}_{\nu]} + M\varepsilon_{[\mu}{}^{ab} \hat{\Omega}_{\nu],b} \\
0 &= D_\mu \hat{B}^a + m\hat{\Omega}_\mu{}^a + M\varepsilon_\mu{}^{ab} \hat{B}_b - \hat{B}_\mu{}^a
\end{aligned}
\tag{24}
$$

The first one appears to be consistent, while the consistency of the second one requires:

$$D_{[\mu}\hat{B}_{\nu]}{}^a = -M\varepsilon_{[\mu}{}^{ab}\hat{B}_{\nu],b} \tag{25}$$

So we choose the following ansatz for the remaining equations $(k \geq 2)$:[5]

$$
\begin{aligned}
0 =\;& D_\mu \hat{B}^{a(k)} - \hat{B}_\mu{}^{a(k)} - A_k \varepsilon_{\mu b}{}^a \hat{B}^{a(k-1)b} \\
& + E_k [e_\mu{}^a \hat{B}^{a(k-1)} - \frac{2}{(2k-1)} g^{a(2)} \hat{B}_\mu{}^{a(k-2)}]
\end{aligned} \tag{26}
$$

where

$$A_2 = \frac{M}{3}, \qquad E_2 = 0$$

The consistency for these equations leads us to the following solution:

$$A_k = \frac{2M}{k(k+1)}, \qquad E_k = \frac{(k^2-4)}{(2k+1)}[\frac{M^2}{k^2} + \Lambda] \tag{27}$$

2 Spin 3

In this section we consider one more concrete example that will illustrate how the very same set of zero forms can describe higher spins. Again we begin with the partially massless case (of maximal depth) and then we will turn to the general massive one.

2.1 Partially massless case

The frame-like gauge invariant description requires three pairs of physical and auxiliary fields $(\Omega_\mu{}^{ab}, \Phi_\mu{}^{ab})$, $(\Omega_\mu{}^a, \Phi_\mu{}^a)$ and (B^a, A_μ). The free Lagrangian has the form:

$$
\begin{aligned}
\mathcal{L}_0 =\;& - \{ {}^{\mu\nu}_{ab} \} \Omega_\mu{}^{ac} \Omega_\nu{}^b{}_c + \varepsilon^{\mu\nu\alpha} \Omega_\mu{}^{ab} D_\nu \Phi_{\alpha,ab} + \frac{1}{2} \{ {}^{\mu\nu}_{ab} \} \Omega_\mu{}^a \Omega_\nu{}^b \\
& - \varepsilon^{\mu\nu\alpha} \Omega_\mu{}^a D_\nu \Phi_{\alpha,a} + \frac{1}{2} B^a B_a - \varepsilon^{\mu\nu\alpha} B_\mu D_\nu A_\alpha \\
& - \varepsilon^{\mu\nu\alpha} [3b_2 \Omega_{\mu,\nu}{}^a \Phi_{\alpha,a} + b_2 \Phi_{\mu,\nu}{}^a \Omega_\alpha{}^a - 2b_1 \Omega_{\mu,\nu} A_\alpha + b_1 \Phi_{\mu,\nu} B_\alpha]
\end{aligned} \tag{28}
$$

where

$$b_2{}^2 = \frac{2\Lambda}{3}, \qquad b_1{}^2 = \frac{4\Lambda}{3}$$

This Lagrangian is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta\Omega_\mu{}^{ab} &= D_\mu \eta^{ab} - \frac{b_2}{2}(e_\mu{}^{(a}\eta^{b)} - \frac{2}{3} g^{ab}\eta_\mu) \\
\delta\Phi_\mu{}^{ab} &= D_\mu \xi^{ab} - \varepsilon_\mu{}^{c(a}\eta^{b)}{}_c - \frac{3b_2}{2}(e_\mu{}^{(a}\xi^{b)} - \frac{2}{3} g^{ab}\xi_\mu) \\
\delta\Omega_\mu{}^a &= D_\mu \eta^a - 3b_2\eta_\mu{}^a \\
\delta\Phi_\mu{}^a &= D_\mu \xi^a + \varepsilon_\mu{}^{ab}\eta_b - b_2\xi_\mu{}^a + 2b_1 e_\mu{}^a\xi \\
\delta B^a &= -2b_1\eta^a, \qquad \delta A_\mu = D_\mu\xi + b_1\xi_\mu
\end{aligned} \tag{29}
$$

---

[5]Note that these equations alone do break parity. The original parity even theory will be restored if we combine both pairs $(\hat{\Omega}, \hat{B})$ and $(\hat{\Phi}, \hat{\pi})$ with the correct coefficients.

Correspondingly, there exist six gauge invariant objects and hence the first six unfolded equations:

$$
\begin{aligned}
0 &= D_{[\mu}\Omega_{\nu]}{}^{ab} - \frac{b_2}{2}(e_{[\mu}{}^{(a}\Omega_{\nu]}{}^{b)} + \frac{2}{3}g^{ab}\Omega_{[\mu,\nu]}) \\
0 &= D_{[\mu}\Phi_{\nu]}{}^{ab} - \varepsilon_{[\mu}{}^{c(a}\Omega_{\nu]}{}^{b)}{}_c - \frac{3b_2}{2}(e_{[\mu}{}^{(a}\Phi_{\nu]}{}^{b)} + \frac{2}{3}g^{ab}\Phi_{[\mu,\nu]}) \\
0 &= D_{[\mu}\Omega_{\nu]}{}^{a} + 3b_2\Omega_{[\mu,\nu]}{}^{a} - b_1 e_{[\mu}{}^{a}B_{\nu]} \\
0 &= D_{[\mu}\Phi_{\nu]}{}^{a} + \varepsilon_{[\mu}{}^{ab}\Omega_{\nu],b} + b_2\Phi_{[\mu,\nu]}{}^{a} + 2b_1 e_{[\mu}{}^{a}A_{\nu]} \\
0 &= D_\mu B^a + 2b_1\Omega_\mu{}^a - B_\mu{}^a \\
0 &= D_{[\mu}A_{\nu]} - \varepsilon_{\mu\nu a}B^a - b_1\Phi_{[\mu,\nu]}
\end{aligned}
\tag{30}
$$

where gauge invariance requires that

$$
\delta B^{ab} = -6b_2 b_1 \eta^{ab}
\tag{31}
$$

All equations except the one for the B^a are already consistent, so we have to deal with the one equation. It looks exactly the same as in the spin-2 case considered above, but the crucial difference is that now the zero-form B^{ab} is not gauge invariant. Indeed, consistency for the B^a equation gives now:

$$
D_{[\mu}B_{\nu]}{}^a = -6b_2 b_1 \Omega_{[\mu,\nu]}{}^a + \frac{5b_2{}^2}{2}e_{[\mu}{}^a B_{\nu]}
\tag{32}
$$

So we take the following form for the next equation:

$$
0 = D_\mu B^{ab} + 6b_2 b_1 \Omega_\mu{}^{ab} - \frac{3b_2{}^2}{2}(e_\mu{}^{(a}B^{b)} - \frac{2}{3}g^{ab}B_\mu) - B_\mu{}^{ab}
\tag{33}
$$

In turn, its consistency leads to

$$
D_{[\mu}B_{\nu]}{}^{a(2)} = 0
\tag{34}
$$

where $B^{a(3)}$ (as well as all $B^{a(k)}$, $k \geq 3$) are gauge invariant. Thus, taking into account that we have parity even theory, we obtain the following equations for all higher rank zero-forms ($k \geq 3$):

$$
0 = D_\mu B^{a(k)} - B_\mu{}^{a(k)} + \frac{(k^2-9)}{(2k+1)}\Lambda[e_\mu{}^a B^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}B_\mu{}^{a(k-2)}]
\tag{35}
$$

2.2   General massive case

This time to simplify presentation from the very beginning we will use the possibility to separate the variables after partial gauge fixing (see appendix C) and consider the subsystem containing the fields $(\hat{\Omega}_\mu{}^{ab}, \hat{\Omega}_\mu{}^a, \hat{B}^a)$ only. The corresponding Lagrangian looks like:

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4M_2}[2M_2\{{}^{\mu\nu}_{ab}\}\hat{\Omega}_\mu{}^{ac}\hat{\Omega}_\nu{}^b{}_c - \varepsilon^{\mu\nu\alpha}\hat{\Omega}_\mu{}^{a(2)}D_\nu\hat{\Omega}_{\alpha,a(2)}] \\
&+\frac{1}{4M_1}[\{{}^{\mu\nu}_{ab}\}\hat{\Omega}_\mu{}^a\hat{\Omega}_\nu{}^b - \varepsilon^{\mu\nu\alpha}\hat{\Omega}_\mu{}^a D_\nu\hat{\Omega}_{\alpha,a}] \\
&+\frac{1}{4M_1}[M_1\hat{B}^a\hat{B}_a + \varepsilon^{\mu\nu\alpha}\hat{B}_\mu D_\nu\hat{B}_\alpha] \\
&+\varepsilon^{\mu\nu\alpha}[-\frac{b_2}{2M_2}\hat{\Omega}_{\mu,\nu}{}^a\hat{\Omega}_{\alpha,a} + \frac{b_1}{2M_1}\hat{\Omega}_{\mu,\nu}\hat{B}_\alpha]
\end{aligned}
\tag{36}
$$

8

where

$$M_2{}^2 = \frac{1}{4}[m^2 - 4\Lambda], \qquad M_1{}^2 = \frac{9}{4}[m^2 - 4\Lambda]$$

$$b_2{}^2 = \frac{m^2}{6}, \qquad b_1{}^2 = \frac{4}{3}[m^2 - 3\Lambda]$$

This Lagrangian is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta\hat{\Omega}_\mu{}^{a(2)} &= D_\mu\hat{\eta}^{a(2)} - \frac{b_2}{2}[e_\mu{}^a\hat{\eta}^a - \frac{2}{3}g^{a(2)}\hat{\eta}_\mu] - M_2\varepsilon_{\mu b}{}^a\hat{\eta}^{ab} \\
\delta\hat{\Omega}_\mu{}^a &= D_\mu\hat{\eta}^a - 3b_2\hat{\eta}_\mu{}^a + M_1\varepsilon_\mu{}^{ab}\hat{\eta}_b \\
\delta\hat{B}^a &= -2b_1\hat{\eta}^a
\end{aligned}
\tag{37}
$$

There exist three gauge invariant objects giving us the first three unfolded equations:

$$
\begin{aligned}
0 &= D_{[\mu}\hat{\Omega}_{\nu]}{}^{a(2)} - \frac{b_2}{2}[e_{[\mu}{}^a\hat{\Omega}_{\nu]}{}^a + \frac{2}{3}g^{a(2)}\hat{\Omega}_{[\mu,\nu]}] - M_2\varepsilon_{[\mu b}{}^a\hat{\Omega}_{\nu]}{}^{ab} \\
0 &= D_{[\mu}\hat{\Omega}_{\nu]}{}^a + 3b_2\hat{\Omega}_{[\mu,\nu]}{}^a + M_1\varepsilon_{[\mu}{}^{ab}\hat{\Omega}_{\nu],b} - b_1 e_{[\mu}{}^a\hat{B}_{\nu]} \\
0 &= D_\mu\hat{B}^a + 2b_1\hat{\Omega}_\mu{}^a + M_1\varepsilon_\mu{}^{ab}\hat{B}_b - \hat{B}_\mu{}^a
\end{aligned}
\tag{38}
$$

where

$$\delta\hat{B}^{a(2)} = -6b_2 b_1\hat{\eta}^{a(2)}$$

The first two are consistent, while for the third one we obtain:

$$D_{[\mu}B_{\nu]}{}^a = -6b_2 b_1\hat{\Omega}_{[\mu,\nu]}{}^a - M_1\varepsilon_{[\mu}{}^{ab}\hat{B}_{\nu],b} + \frac{5b_2{}^2}{2}e_{[\mu}{}^a\hat{B}_{\nu]} \tag{39}$$

So we take the following form for the next equation:

$$0 = D_\mu\hat{B}^{a(2)} + 6b_2 b_1\hat{\Omega}_\mu{}^{a(2)} - M_2\varepsilon_{\mu b}{}^a\hat{B}^{ab} - \frac{3b_2{}^2}{2}[e_\mu{}^a\hat{B}^a - \frac{2}{3}g^{a(2)}\hat{B}_\mu] - \hat{B}_\mu{}^{a(2)} \tag{40}$$

In turn, the consistency for the last equations gives

$$D_{[\mu}\hat{B}_{\nu]}{}^{a(2)} = M_2\varepsilon_{[\mu b}{}^a\hat{B}_{\nu]}{}^a \tag{41}$$

Taking into account that all $\hat{B}^{a(k)}$, $k \geq 3$ are gauge invariant, we take the following ansatz for the remaining equations:

$$
\begin{aligned}
0 &= D_\mu\hat{B}^{a(k)} - \hat{B}_\mu{}^{a(k)} - A_k\varepsilon_{\mu b}{}^a\hat{B}^{a(k-1)b} \\
&\quad + E_k[e_\mu{}^a\hat{B}^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\hat{B}_\mu{}^{a(k-2)}]
\end{aligned}
\tag{42}
$$

where

$$A_3 = \frac{M_2}{2}, \qquad E_3 = 0$$

The consistency of these equations leads to the following solution:

$$A_k = \frac{6M_2}{k(k+1)}, \qquad E_k = \frac{(k^2-9)}{(2k+1)}[\frac{4M_2{}^2}{k^2} + \Lambda] \tag{43}$$

9

3 Arbitrary spin

Now we are ready to consider generalization to the case of arbitrary spin. Once again, a separate subsection will be devoted to the partially massless case of the maximal depth.

3.1 Partially massless case

The Lagrangian, gauge transformations and the whole set of gauge invariant objects are given in the appendix B. From these formulas one can see that auxiliary fields $\Omega_\mu{}^{a(k)}$ and B^a generate a closed subsystem in a sense that they transform non-trivially under the η transformations only and as a result their gauge invariant objects contain only the auxiliary fields themselves. Thus we begin with equations for the auxiliary fields (one can easily check that the equations for the physical ones are consistent):

$$
\begin{aligned}
0 &= D_{[\mu}\Omega_{\nu]}{}^{a(k)} + \frac{(k+2)b_{k+1}}{k}\Omega_{[\mu,\nu]}{}^{a(k)} \\
&\quad - \frac{b_k}{k}[e_{[\mu}{}^a\Omega_{\nu]}{}^{a(k-2)} + \frac{2}{(2k-1)}g^{a(2)}\Omega_{[\mu,\nu]}{}^{a(k-2)}] \\
0 &= D_{[\mu}\Omega_{\nu]}{}^a + 3b_2\Omega_{[\mu,\nu]}{}^a - 2b_1{}^2 e_{[\mu}{}^a B_{\nu]} \\
0 &= D_\mu B^a + \Omega_\mu{}^a - B_\mu{}^a
\end{aligned}
\tag{44}
$$

where parameters b_k are given in (68) and to simplify subsequent formulas we have changed normalization for the zero forms. We will also need the gauge transformations that look like:

$$
\begin{aligned}
\delta\Omega_\mu{}^{a(k)} &= D_\mu\eta^{a(k)} - \frac{(k+2)b_{k+1}}{k}\eta_\mu{}^{a(k)} \\
&\quad - \frac{b_k}{k}(e_\mu{}^a\eta^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\eta_\mu{}^{a(k-2)}) \\
\delta\Omega_\mu{}^a &= D_\mu\eta^a - 3b_2\eta_\mu{}^a, \qquad \delta B^a = -\eta^a, \qquad \delta B^{a(2)} = -\eta^{a(2)}
\end{aligned}
\tag{45}
$$

All equations except the last one are consistent, while for the last one we obtain:

$$
D_{[\mu}B_{\nu]}{}^a = -\Omega_{[\mu,\nu]}{}^a + \frac{5b_2}{6}e_{[\mu}{}^a B_{\nu]}
\tag{46}
$$

Thus the next equation looks like

$$
0 = D_\mu B^{a(2)} + \Omega_\mu{}^{a(2)} - \frac{b_2}{2}[e_\mu{}^a - \frac{2}{3}g^{a(2)}B_\mu] - 2b_3 B_\mu{}^{a(2)}
\tag{47}
$$

Till now all looks exactly as in the spin-3 case, but now gauge invariance requires that

$$
\delta B^{a(3)} = -\eta^{a(3)}
$$

so the equation for this zero-form also must contain one-forms and so on. Thus let us consider the chain of equations ($2 \leq k \leq s-2$):

$$
0 = D_\mu B^{a(k)} + \Omega_\mu{}^{a(k)} - \frac{(k+2)b_{k+1}}{k}B_\mu{}^{a(k)}
$$

$$-\frac{b_k}{k}[e_\mu{}^a B^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}B_\mu{}^{a(k-2)}] \tag{48}$$

$$0 = D_\mu B^{a(s-1)} + \Omega_\mu{}^{a(s-1)} - B_\mu{}^{a(s-1)}$$
$$-\frac{b_{s-1}}{(s-1)}[e_\mu{}^a B^{a(s-2)} - \frac{2}{(2s-3)}g^{a(2)}B_\mu{}^{a(s-2)}] \tag{49}$$

where gauge invariance is achieved provided

$$\delta B^{a(k)} = -\eta^{a(k)}, \quad 2 \le k \le s-1, \quad \delta B^{a(s)} = 0 \tag{50}$$

All the equations (48) are consistent, while the consistency of (49) gives:

$$D_{[\mu}B_{\nu]}{}^{a(s-1)} = 0 \tag{51}$$

All zero-forms $B^{a(k)}$ with $k \ge s$ are gauge invariant, so we obtain all the remaining equations $k \ge s$:

$$0 = D_\mu B^{a(k)} - B_\mu{}^{a(k)} + \frac{(k^2 - s^2)}{(2k+1)}\Lambda[e_\mu{}^a B^{a(k-1)} - \frac{2}{(2k-1)}B_\mu{}^{a(k-2)}] \tag{52}$$

3.2 General massive case

Similarly to the spin-3 case from the very beginning we will use partially gauge fixed version with separated variables and consider the subsystem containing fields $\hat\Omega$ and $\hat B$ only. The Lagrangian and the whole set of gauge invariant objects are given in the appendix C and here we begin with the first set of unfolded equations (with the changed normalization for the zero forms):

$$0 = D_{[\mu}\hat\Omega_{\nu]}{}^{a(k)} - M_k\varepsilon_{[\mu b}{}^a\hat\Omega_{\nu]}{}^{a(k-1)b} + \frac{(k+2)b_{k+1}}{k}\hat\Omega_{[\mu,\nu]}{}^{a(k)}$$
$$-\frac{b_k}{k}[e_{[\mu}{}^a\hat\Omega_{\nu]}{}^{a(k-1)} + \frac{2}{(2k-1)}g^{a(2)}\hat\Omega_{[\mu,\nu]}{}^{a(k-2)}]$$
$$0 = D_{[\mu}\hat\Omega_{\nu]}{}^a + M_1\varepsilon_{[\mu}{}^{ab}\hat\Omega_{\nu],b} + 3b_2\hat\Omega_{[\mu,\nu]}{}^a - 2b_1{}^2 e_{[\mu}{}^a\hat B_{\nu]} \tag{53}$$
$$0 = D_\mu\hat B^a + \hat\Omega_\mu{}^a + M_1\varepsilon_\mu{}^{ab}\hat B_b - 3b_2\hat B_\mu{}^a$$

where parameters b_k and M_k are given in appendix A. We will also need the gauge transformations:

$$\delta\hat\Omega_\mu{}^{a(k)} = D_\mu\hat\eta^{a(k)} - M_k\varepsilon_{[\mu b}{}^a\hat\eta^{a(k-1)b} - \frac{(k+2)b_{k+1}}{k}\hat\eta_\mu{}^{a(k)}$$
$$-\frac{b_k}{k}[e_\mu{}^a\hat\eta^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\eta_\mu{}^{a(k-2)}]$$
$$\delta\hat\Omega_\mu{}^a = D_\mu\hat\eta^a + M_1\varepsilon_\mu{}^{ab}\hat\eta_b - 3b_2\hat\eta_\mu{}^a \tag{54}$$
$$\delta\hat B^a = -\hat\eta^a, \qquad \delta B^{a(2)} = -\hat\eta^{a(2)}$$

All equations except the last one are consistent, while for the last one we obtain:

$$D_{[\mu}\hat B_{\nu]}{}^a = -\hat\Omega_{[\mu,\nu]}{}^a - M_1\varepsilon_{[\mu}{}^{ab}\hat B_{\nu],b} + \frac{5b_2}{6}e_{[\mu}{}^a\hat B_{\nu]} \tag{55}$$

11

Thus the next equation has the form:

$$0 = D_\mu \hat{B}^{a(2)} + \hat{\Omega}_\mu{}^{a(2)} - M_2 \varepsilon_{\mu b}{}^a \hat{B}^{ab} - \frac{b_2}{2}[e_\mu{}^a \hat{B}^a - \frac{2}{3}g^{a(2)}\hat{B}_\mu] - 2b_3 \hat{B}_\mu{}^{a(2)} \qquad (56)$$

where $\delta \hat{B}^{a(3)} = -\hat{\eta}^{a(3)}$ and so on. As in the partially massless case we proceed with the whole set of equations

$$
\begin{aligned}
0 &= D_\mu \hat{B}^{a(k)} + \hat{\Omega}_\mu{}^{a(k)} - M_k \varepsilon_{\mu b}{}^a \hat{B}^{a(k-1)b} - \frac{(k+2)b_{k+1}}{k}\hat{B}_\mu{}^{a(k)} \\
&\quad - \frac{b_k}{k}[e_\mu{}^a \hat{B}^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\hat{B}_\mu{}^{a(k-2)}], \qquad 2 \le k \le s-2 \qquad (57) \\
0 &= D_\mu \hat{B}^{a(s-1)} + \hat{\Omega}_\mu{}^{a(s-1)} - M_{s-1}\varepsilon_{\mu b}{}^a \hat{B}^{a(s-2)b} \\
&\quad - \frac{b_{s-1}}{(s-1)}[e_\mu{}^a \hat{B}^{a(s-2)} - \frac{2}{(2s-3)}g^{a(2)}\hat{B}_\mu{}^{a(s-3)}] - \hat{B}_\mu{}^{a(s-1)} \qquad (58)
\end{aligned}
$$

where

$$\delta \hat{B}^{a(k)} = -\hat{\eta}^{a(k)}, \quad 2 \le k \le s-1, \qquad \delta \hat{B}^{a(s)} = 0 \qquad (59)$$

All equations (57) are consistent, while the consistency for the (58) gives:

$$D_{[\mu}\hat{B}_{\nu]}{}^{a(s-1)} = M_{s-1}\varepsilon_{[\mu b}{}^a \hat{B}_{\nu]}{}^{a(s-2)b} \qquad (60)$$

Finally taking into account that all $\hat{B}^{a(k)}$ with $k \ge s$ are gauge invariant we obtain all the remaining equations:

$$
\begin{aligned}
0 &= D_\mu \hat{B}^{a(k)} - \hat{B}_\mu{}^{a(k)} - A_k \varepsilon_{\mu b}{}^a \hat{B}^{a(k-1)b} \\
&\quad + E_k[e_\mu{}^a \hat{B}^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\hat{B}_\mu{}^{a(k-2)}] \qquad (61)
\end{aligned}
$$

$$A_k = \frac{s(s-1)}{k(k+1)}M_{s-1}, \qquad E_k = \frac{(k^2-s^2)}{(2k+1)}[\frac{(s-1)^2 M_{s-1}{}^2}{k^2} + \Lambda] \qquad (62)$$

Summary

Thus we have constructed the unfolded formulation for the massive bosonic higher spins in three dimensions as well as for their partially massless limits of the maximal depth. In spite of the number of features specific for three dimensions, the general picture appears to be very much like in $d \ge 4$ case [12]. Namely, we have a finite number of zero forms that are not gauge invariant, transform as the Stueckelberg fields and have equations containing both one-forms and zero-forms. Besides, we have an infinite number of the gauge invariant zero-forms those equations contain only zero-forms themselves. As it was expected, the unfolded formalism in $d = 3$ turns out to be much simpler then the one for $d \ge 4$ so one can hope that such formalism could be useful for the investigations of possible interactions.

Acknowledgments

A Gauge invariant description of massive higher spins

In three dimensions the frame-like gauge invariant description for the massive arbitrary spin s [10] requires introduction of the following set of physical and auxiliary fields: $(\Omega_\mu{}^{a(k)}, \Phi_\mu{}^{a(k)})$, $1 \le k \le s-1$, (B^a, A_μ) and (π^a, φ), where $\Omega_\mu{}^{a(k)}$ and $\Phi_\mu{}^{a(k)}$ are symmetric and traceless on their local indices. The whole Lagrangian consists of the three parts:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

$$
\begin{aligned}
\mathcal{L}_0 &= \sum_{k=1}^{s-1}(-1)^{k+1}[\frac{k}{2}\{{}^{\mu\nu}_{ab}\}\Omega_\mu{}^{ac(k-1)}\Omega_\nu{}^b{}_{c(k-1)} - \varepsilon^{\mu\nu\alpha}\Omega_\mu{}^{a(k)}D_\nu\Phi_{\alpha,a(k)} \\
&\quad + \frac{1}{2}B^a B_a - \varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha - \frac{1}{2}\pi^a\pi_a + \pi^\mu D_\mu\varphi & (63)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_1 &= \sum_{k=2}^{s-1}(-1)^{k+1}b_k\varepsilon^{\mu\nu\alpha}[\frac{(k+1)}{(k-1)}\Omega_{\mu,\nu}{}^{a(k-1)}\Phi_{\alpha,a(k-1)} + \Omega_{\mu,a(k-1)}\Phi_{\nu,\alpha}{}^{a(k-1)}] \\
&\quad - b_1\varepsilon^{\mu\nu\alpha}[2\Omega_{\mu,\nu}A_\alpha - B_\mu\Phi_{\nu,\alpha}] + 2M_1\pi^\mu A_\mu & (64)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_2 &= \sum_{k=1}^{s-1}(-1)^{k+1}\frac{kM_k{}^2}{2}\{{}^{\mu\nu}_{ab}\}\Phi_\mu{}^{ac(k-1)}\Phi_\nu{}^b{}_{c(k-1)} \\
&\quad + 2M_1 b_1 e^\mu{}_a\Phi_\mu{}^a\varphi + 3b_1{}^2\varphi^2 & (65)
\end{aligned}
$$

where $\mathcal{L}_0$ and $\mathcal{L}_2$ contain kinetic and mass-like terms for all fields while $\mathcal{L}_1$ contains cross-terms gluing all these fields together. Here:

$$b_k{}^2 = \frac{(k-1)(s-k)(s+k)}{k(k+1)(2k+1)}[m^2 - (s-k-1)(s+k-1)\Lambda], \qquad k \ge 2$$

$$b_1{}^2 = \frac{(s-1)(s+1)}{6}[m^2 - s(s-2)\Lambda]$$

$$M_k{}^2 = \frac{s^2}{k^2(k+1)^2}[m^2 - (s-1)^2\Lambda]$$

This Lagrangian is invariant under the following gauge transformations:

$$
\begin{aligned}
\delta\Omega_\mu{}^{a(k)} &= D_\mu\eta^{a(k)} - \frac{(k+2)b_{k+1}}{k}\eta_\mu{}^{a(k)} - M_k{}^2\varepsilon_{\mu b}{}^a\eta^{a(k-1)b} \\
&\quad - \frac{b_k}{k}[e_\mu{}^a\eta^{a(k-1)} - \frac{2}{(2k-1)}g^{a(2)}\eta_\mu{}^{a(k-2)}] \\
\delta\Phi_\mu{}^{a(k)} &= D_\mu\xi^{a(k)} - b_{k+1}\xi_\mu{}^{a(k)} - \varepsilon_{\mu b}{}^a\eta^{a(k-1)b} \\
&\quad - \frac{(k+1)b_k}{k(k-1)}[e_\mu{}^a\xi^{a(k-1)} - \frac{2}{2k-1}g^{a(2)}\xi_\mu{}^{a(k-2)}] \\
\delta\Omega_\mu{}^a &= D_\mu\eta^a - 3b_2\eta_\mu{}^a + M_1{}^2\varepsilon_\mu{}^{ab}\xi_b \\
\delta\Phi_\mu{}^a &= D_\mu\xi^a + \varepsilon_{[\mu}{}^{ab}\eta_b - b_2\xi_\mu{}^a + 2b_1 e_\mu{}^a\xi \\
\delta B^a &= -2b_1\eta^a, \qquad \delta A_\mu = D_\mu\xi + b_1\xi_\mu \\
\delta\pi^a &= 2M_1 b_1\xi^a, \qquad \delta\varphi = 2M_1\xi
\end{aligned}
$$

where the gauge parameters $\eta^{a(k)}$ and $\xi^{a(k)}$ are also symmetric and traceless.

B Partially massless limit

From the explicit formulas given in the previous appendix, one can see that in de-Sitter space $\Lambda > 0$ there exists a number of special mass values where one of the parameters $b_l = 0$. In this case the whole system decomposes into two independent subsystems. The first one with the fields $(\Omega_\mu{}^{a(k)}, \Phi_\mu{}^{a(k)})$, $l \le k \le s-1$, describes the so called partially massless theory, while the remaining fields gives massive spin-l. In three dimensions most of such partially massless fields do not have any physical degrees of freedom and do not require introduction of any zero-forms. The only case with one physical degree of freedom corresponds to

$$M_1 = 0 \quad \Rightarrow \quad m^2 = (s-1)^2\Lambda \tag{66}$$

when spin-0 decouples. Note that in this case all $M_k = 0$ so that the Lagrangian and gauge transformations are greatly simplified:

$$
\begin{aligned}
\mathcal{L}_0 =& \sum_{k=1}^{s-1}(-1)^{k+1}[\frac{k}{2}\{{}^{\mu\nu}_{ab}\}\,\Omega_\mu{}^{ac(k-1)}\Omega_\nu{}^b{}_{c(k-1)} - \varepsilon^{\mu\nu\alpha}\Omega_\mu{}^{a(k)}D_\nu\Phi_{\alpha,a(k)}] \\
&+ \sum_{k=2}^{s-1}(-1)^{k+1}b_k\varepsilon^{\mu\nu\alpha}[\frac{(k+1)}{(k-1)}\Omega_{\mu,\nu}{}^{a(k-1)}\Phi_{\alpha,a(k-1)} + \Omega_{\mu,a(k-1)}\Phi_{\nu,\alpha}{}^{a(k-1)}] \\
&+ \frac{1}{2}B^aB_a - \varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha + b_1\varepsilon^{\mu\nu\alpha}[-2\Omega_{\mu,\nu}A_\alpha + B_\mu\Phi_{\nu,\alpha}]
\end{aligned} \tag{67}
$$

where now

$$b_k{}^2 = \frac{k(k-1)(s^2-k^2)}{(k+1)(2k+1)}\Lambda, \qquad b_1{}^2 = \frac{(s^2-1)}{6}\Lambda \tag{68}$$

$$
\begin{aligned}
\delta\Omega_\mu{}^{a(k)} &= D_\mu\eta^{a(k)} - \frac{(k+2)b_{k+1}}{k}\eta_\mu{}^{a(k)} - \frac{b_k}{k}[e_\mu{}^a\eta^{a(k-1)} - Tr] \\
\delta\Phi_\mu{}^{a(k)} &= D_\mu\xi^{a(k)} - \varepsilon_{\mu b}{}^a\eta^{a(k-1)b} - b_{k+1}\xi_\mu{}^{a(k)} - \frac{(k+1)b_k}{k(k-1)}[e_\mu{}^a\xi^{a(k-1)} - Tr] \\
\delta\Omega_\mu{}^a &= D_\mu\eta^a - 3b_2\eta_\mu{}^a \\
\delta\Phi_\mu{}^a &= D_\mu\xi^a - b_2\xi_\mu{}^a + \varepsilon_\mu{}^{ab}\eta_b + 2b_1e_\mu{}^a\xi \\
\delta B^a &= -2b_1\eta^a, \qquad \delta A_\mu = D_\mu\xi + b_1\xi_\mu
\end{aligned} \tag{69}
$$

As usual in the frame-like gauge invariant formalism, for each field (both physical as well as auxiliary one) one can construct corresponding gauge invariant object:

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{a(k)} &= D_{[\mu}\Omega_{\nu]}{}^{a(k)} + \frac{(k+2)b_{k+1}}{k}\Omega_{[\mu,\nu]}{}^{a(k)} - \frac{b_k}{k}[e_{[\mu}{}^a\Omega_{\nu]}{}^{a(k-1)} - Tr] \\
\mathcal{T}_{\mu\nu}{}^{a(k)} &= D_{[\mu}\Phi_{\nu]}{}^{a(k)} - \varepsilon_{[\mu b}{}^a\Omega_{\nu]}{}^{a(k-1)b} + b_{k+1}\Phi_{[\mu,\nu]}{}^{a(k)} \\
&\quad - \frac{(k+1)b_k}{k(k-1)}[e_{[\mu}{}^a\Phi_{\nu]}{}^{a(k-1)} - Tr] \\
\mathcal{R}_{\mu\nu}{}^a &= D_{[\mu}\Omega_{\nu]}{}^a + 3b_2\Omega_{[\mu,\nu]}{}^a - b_1e_{[\mu}{}^aB_{\nu]} \\
\mathcal{T}_{\mu\nu}{}^a &= D_{[\mu}\Phi_{\nu]}{}^a + \varepsilon_{[\mu}{}^{ab}\Omega_{\nu],b} + b_2\Phi_{[\mu,\nu]}{}^a + 2b_1e_{[\mu}{}^aA_{\nu]} \\
\mathcal{B}_\mu{}^a &= D_\mu B^a + 2b_1\Omega_\mu{}^a - B_\mu{}^a \\
\mathcal{A}_{\mu\nu} &= D_{[\mu}A_{\nu]} - \varepsilon_{\mu\nu a}B^a - b_1\Phi_{[\mu,\nu]}
\end{aligned} \tag{70}
$$

where the gauge invariance requires the introduction of the zero-form B^{ab} which does not enter the free Lagrangian and transforms non-trivially under the η^{ab} transformation:

$$\delta B^{ab} = -6b_2 b_1 \eta^{ab} \tag{71}$$

C   Partial gauge fixing

As is well known [13, 14] (see also [15] and references therein) in the frame-like formalism for the massless higher spin fields in the three dimensional anti-de Sitter space one can introduce combinations of physical and auxiliary fields such that the whole theory (not only the free theory but an interacting one as well) decomposes into two independent subsystems. It was shown in [10] that such separation works for the massive higher spins as well provided one uses a partial gauge fixing to remove scalar field. Moreover, as the frame-like gauge invariant description for massive fields itself, such mechanism works not only in anti-de Sitter space but in Minkowski and de Sitter spaces provided $m^2 > (s-1)^2 \Lambda$.

Let us partially gauge fix the general massive theory described in appendix A by setting the gauge $\varphi = 0$, solve the constraint $A_\mu = \pi_\mu/(2M_1)$ and re-scale $\pi^a \Rightarrow 2M_1 \pi^a$ (taking into account that π^a will play now the role of physical field and not that of the auxiliary one). Resulting Lagrangian takes the form:

$$
\begin{aligned}
\mathcal{L} &= \sum_{k=1}^{s-1} (-1)^{k+1} [\frac{k}{2} \{ {}^{\mu\nu}_{ab} \} \Omega_\mu{}^{ac(k-1)} \Omega_\nu{}^b{}_{c(k-1)} - \varepsilon^{\mu\nu\alpha} \Omega_\mu{}^{a(k)} D_\nu \Phi_{\alpha,a(k)} \\
&\quad + \sum_{k=2}^{s-1} (-1)^{k+1} b_k \varepsilon^{\mu\nu\alpha} [\frac{(k+1)}{(k-1)} \Omega_{\mu,\nu}{}^{a(k-1)} \Phi_{\alpha,a(k-1)} + \Omega_{\mu,a(k-1)} \Phi_{\nu,\alpha}{}^{a(k-1)}] \\
&\quad + \frac{1}{2} B^a B_a - \varepsilon^{\mu\nu\alpha} B_\mu D_\nu \pi_\alpha - b_1 \varepsilon^{\mu\nu\alpha} [2\Omega_{\mu,\nu} \pi_\alpha - B_\mu \Phi_{\nu,\alpha}] \\
&\quad + \sum_{k=1}^{s-1} (-1)^{k+1} \frac{k M_k{}^2}{2} \{ {}^{\mu\nu}_{ab} \} \Phi_\mu{}^{ac(k-1)} \Phi_\nu{}^b{}_{c(k-1)} + 2M_1{}^2 \pi^a \pi_a
\end{aligned} \tag{72}
$$

Let us introduce new variables:

$$\hat{\Omega}_\mu{}^{a(k)} = \Omega_\mu{}^{a(k)} + M_k \Phi_\mu{}^{a(k)}, \qquad \hat{\Phi}_\mu{}^{a(k)} = \Omega_\mu{}^{a(k)} - M_k \Phi_\mu{}^{a(k)} \tag{73}$$

$$\hat{B}^a = B^a - 2M_1 \pi^a, \qquad \hat{\pi}^a = B^a + 2M_1 \pi^a \tag{74}$$

Then the whole Lagrangian can be rewritten as:

$$\mathcal{L} = \mathcal{L}(\hat{\Omega}, \hat{B}) - \mathcal{L}(\hat{\Phi}, \hat{\pi})$$

where, for example,

$$
\begin{aligned}
\mathcal{L}(\hat{\Omega}, \hat{B}) &= \sum_{k=1}^{s-1} \frac{(-1)^{k+1}}{4M_k} [k M_k \{ {}^{\mu\nu}_{ab} \} \hat{\Omega}_\mu{}^{ac(k-1)} \hat{\Omega}_\nu{}^b{}_{c(k-1)} - \varepsilon^{\mu\nu\alpha} \hat{\Omega}_\mu{}^{a(k)} D_\nu \hat{\Omega}_{\alpha,a(k)}] \\
&\quad + \frac{1}{4M_1} [M_1 \hat{B}^a \hat{B}_a + \varepsilon^{\mu\nu\alpha} \hat{B}_\mu D_\nu \hat{B}_\alpha] \\
&\quad + \sum_{k=2}^{s-1} \frac{(-1)^{k+1} b_k}{2M_k} \varepsilon^{\mu\nu\alpha} \hat{\Omega}_{\mu,\nu}{}^{a(k-1)} \hat{\Omega}_{\alpha,a(k-1)} + \frac{b_1}{2M_1} \varepsilon^{\mu\nu\alpha} \hat{\Omega}_{\mu,\nu} \hat{B}_\alpha
\end{aligned} \tag{75}
$$

15

This Lagrangian is invariant under the following gauge transformations;

$$
\begin{aligned}
\delta\hat{\Omega}_\mu{}^{a(k)} &= D_\mu\hat{\eta}^{a(k)} - M_k\varepsilon_{\mu b}{}^a\hat{\eta}^{a(k-1)b} - \frac{(k+2)b_{k+1}}{k}\hat{\eta}_\mu{}^{a(k)} \\
&\quad - \frac{b_k}{k}[e_\mu{}^a\hat{\eta}^{a(k-1)} - Tr] \\
\delta\hat{\Omega}_\mu{}^a &= D_\mu\hat{\eta}^a + M_1\varepsilon_\mu{}^{ab}\hat{\eta}_b - 3b_2\hat{\eta}_\mu{}^a \\
\delta\hat{B}^a &= -2b_1\hat{\eta}^a
\end{aligned}
\tag{76}
$$

where

$$
\hat{\eta}^{a(k)} = \eta^{a(k)} + M_k\xi^{a(k)}
\tag{77}
$$

Moreover, for each field we can still construct corresponding gauge invariant object:

$$
\begin{aligned}
\hat{\mathcal{F}}_{\mu\nu}{}^{a(k)} &= D_{[\mu}\hat{\Omega}_{\nu]}{}^{a(k)} - M_k\varepsilon_{[\mu b}{}^a\hat{\Omega}_{\nu]}{}^{a(k-1)b} + \frac{(k+2)b_{k+1}}{k}\hat{\Omega}_{[\mu,\nu]}{}^{a(k)} \\
&\quad - \frac{b_k}{k}[e_{[\mu}{}^a\hat{\Omega}_{\nu]}{}^{a(k-1)} - Tr] \\
\hat{\mathcal{F}}_{\mu\nu}{}^a &= D_{[\mu}\hat{\Omega}_{\nu]}{}^a + M_1\varepsilon_{[\mu}{}^{ab}\hat{\Omega}_{\nu],b} + 3b_2\hat{\Omega}_{[\mu,\nu]}{}^a - b_1 e_{[\mu}{}^a\hat{B}_{\nu]} \\
\hat{\mathcal{B}}_\mu{}^a &= D_\mu\hat{B}^a + 2b_1\hat{\Omega}_\mu{}^a + M_1\varepsilon_\mu{}^{ab}\hat{B}_b - \hat{B}_\mu{}^a
\end{aligned}
\tag{78}
$$

where, similarly to the partially massless case, gauge invariance requires introduction of the zero-form $\hat{B}^{ab}$ such that

$$
\delta\hat{B}^{ab} = -6b_1 b_2\hat{\eta}^{ab}
\tag{79}
$$

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